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Polynomial Fuzzy-Model-Based Control Systems: Stability Analysis via Approximated Membership Functions Considering Sector Nonlinearity of Control Input

H.K. Lam *Senior Member, IEEE*, Chuang Liu, Ligang Wu *Senior Member, IEEE* and Xudong Zhao *Member, IEEE*

Abstract—This paper presents the stability analysis of polynomial fuzzy-model-based (PFMB) control systems of which both the polynomial fuzzy model and the polynomial fuzzy controller are allowed to have their own set of premise membership functions. In order to address the input nonlinearity, the control signal is considered to be bounded by a sector with nonlinear bounds. These nonlinear lower and upper bounds of the sector are constructed by combining local bounds using fuzzy blending such that local information of input nonlinearity can be taken into account. With the consideration of imperfectly matched membership functions and input nonlinearity, the applicability of the PFMB control scheme can be further enhanced. To facilitate the stability analysis, a general form of approximated membership functions representing the original ones is introduced. As a result, approximated membership functions can be brought into the stability analysis leading to relaxed stability conditions. Sum of squares (SOS) approach is employed to obtain the stability conditions based on Lyapunov stability theory. Simulation examples are presented to demonstrate the feasibility of the proposed method.

Index Terms—Polynomial fuzzy-model-based (PFMB) control systems, stability analysis, Taylor series membership functions (TSMFs), sector nonlinearity of control input, sum of squares (SOS).

I. INTRODUCTION

TAKAGI-Sugeno (T-S) fuzzy model [1], [2] provides a mathematical way to accurately represent nonlinear systems in a general form in favor of the stability analysis. Based on the T-S fuzzy model, systematic stability analysis was carried out via the Lyapunov stability theory [3], achieving a set of stability conditions in terms of linear matrix inequalities (LMIs) [4], [5] to guarantee the stability/stabilization of

nonlinear systems. Since then, many methods of relaxing LMI-based stability conditions have been proposed [6], [7] and then generalized by Pólya's theory [8], [9]. To further relax stability conditions, two main areas were considered: one is to adopt more complicated Lyapunov function candidates such as piecewise linear Lyapunov function [10], [11], fuzzy Lyapunov function [12]–[14], and switching Lyapunov function [15], [16]; another one is exploiting the property of membership functions, which includes the information of membership functions to stability analysis [13], [17]–[28]. The relaxation techniques contain the concept of parallel distributed compensation (PDC) and slack matrices through S-procedure [29]. Despite of the research on stability analysis, the concept of fuzzy control was extended to other control problems such as output feedback control [30]–[32].

Recently, the T-S fuzzy-model-based (FMB) control was generalized to the polynomial fuzzy-model-based (PFMB) control [33], which allows polynomials to exist in the PFMB control system. The sector nonlinearity technique for the construction of T-S fuzzy model was also extended based on Taylor series expansion [34]. The stability conditions are in terms of sum of squares (SOS) [35] rather than LMI. In [36], it has been demonstrated that the SOS design approach based on PFMB control system is better than the LMI design approach based on T-S FMB control system in the sense that the guaranteed cost in the SOS approach is lower than that in the LMI approach. The comparison was carried out on a micro helicopter to make it follow a target trajectory. In the SOS approach, the polynomial feedback gains are set to be a function of system states, which cannot be implemented by LMI approach and thus leads to the superiority of SOS approach. Although PFMB control systems can potentially provide more relaxed stability conditions than T-S FMB control systems [33], the information of the shape of membership functions still need to be considered to further relax the conservativeness of stability conditions. Since polynomials can be handled in stability conditions, they can be exploited to approximate membership functions such that the information is brought into stability conditions. In [21], polynomial membership functions were proposed to approximate the original membership functions in each operating subdomain. Whereas, there was no systematic way to determine approximated membership functions. In [24], a piecewise linear membership function (PLMF) was pro-

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H.K. Lam and Chuang Liu are with the Department of Informatics, King's College London, Strand, London, WC2R 2LC, United Kingdom. e-mail: {hak-keung.lam, chuang.liu}@kcl.ac.uk.

L. Wu is with Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin, 150001, China. e-mail: ligangwu@hit.edu.cn.

X. Zhao is with the College of Engineering, Bohai University, Jinzhou 121013, China. e-mail: xdzhaohit@gmail.com

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posed to achieve the approximation of membership functions in stability analysis. Nonetheless, the approximation errors need to be improved due to the limited expression capability of the linear functions. Consequently, a systematic method of approximating membership functions is required for reducing the conservativeness of stability conditions. In addition, the switching polynomial Lyapunov function was employed to replace the quadratic Lyapunov function in stability analysis [27], which also offered more relaxed stability conditions.

In real control systems, actuators have nonlinearities caused by physical constraints or technological factors [37]. Ignoring these nonlinearities may lead to degradation, instability and damage of systems. As a result, the input nonlinearities have been widely investigated in the past decades. The sector nonlinearity of control input was proposed to formulate nonlinear features of actuators [38]. However, the sector can only be constructed by two straight zeroaxial lines. As a result, some other nonlinearities such as deadzone and saturation [39] cannot be described by sector nonlinearity. Hence, a generalized sector condition was presented to cover the characteristics of deadzone and saturation [37]. In [40]–[42], the input nonlinearity containing sector nonlinearity and deadzone was investigated, and so-called gain reduction tolerances can even be unknown constants. Furthermore, the input nonlinearity model, which comprises sector nonlinearity, deadzone, and saturation, was presented in [43]. Additionally, other types of general sector nonlinearity and decentralized sector nonlinearity were proposed in [44], [45]. In [45], the lower and upper bounds of the sector in the first quadrant are allowed to be different from those in the third quadrants. A similar property to sector nonlinearity called the loss of actuator effectiveness was investigated [46], which is one type of actuator faults and is also described by the known constant lower and upper bounds. To the best of authors' knowledge, the lower and upper bounds of the sector are usually linear considered in the existing work. Therefore, the constructed sector may poorly approximate the actual input nonlinearity leading to conservative stability conditions. More precise approximation of the sector can be achieved by considering nonlinear bounds of sector, which can describe the specific input nonlinearity better than using the sector with only linear bounds. Furthermore, allowing bounds of the sector to be varied with both system states and control signals rather than constants helps to describe a wider range of input nonlinearity.

With regard to the systems used for investigating input nonlinearity, linear systems were studied in [37]. Then a system consisting of a linear system and a time-varying nonlinear element in the feedback connection was investigated [44], [47]. More complex systems were considered including uncertain time-delayed systems [40], [48], uncertain chaotic systems [41], [42], [45], flexible air-breathing hypersonic vehicles [43] and near-space vehicles [46]. In [43], [46], T-S fuzzy model was employed to represent these known nonlinear systems. Nonetheless, those systems are specific systems rather than a general form, and the polynomial fuzzy model which is the extension of T-S fuzzy model has not been considered. It motivates the investigation on the PFMB control systems with nonlinear sector of control input.

In this paper, we investigate the stability issues of PFMB control systems, aiming to relax the stability conditions. To achieve this goal, some information needs to be brought from the membership functions into the stability analysis. Since polynomials can be handled in SOS conditions, we approximate the original membership functions by combining local polynomials such that membership functions can be taken into account in stability analysis with the consideration of approximation error. Unlike the existing works stated above, we aim to present a systematic representation approximating the original membership functions and facilitating the stability analysis. Taylor series expansion, but not limited to, naturally becomes the expected method for the approximation. This method not only yields polynomials for approximation, but also offers the truncation order and expansion points to be determined by users. By using Taylor series membership functions (TSMFs), the conservativeness of stability conditions can be progressively reduced with the order of Taylor series and density of expansion points increasing. This way of approximation is more general and organized than existing work. Meanwhile, we aim to deal with the problem of sector nonlinearity of control input. To draw a distinction from previous work, we consider nonlinear system represented by a polynomial fuzzy system with the sector of nonlinear bounds characterized by both system states and control signals. As the input nonlinearity complicates the system dynamics, it makes the analysis a challenging task. In consideration of the powerful ability of representing nonlinear systems, the fuzzy modeling approach is employed to construct both the nonlinear system and the nonlinear bounds of the sector. Following the polynomial fuzzy model, SOS-based stability conditions are obtained based on Lyapunov stability theory. The above improvements on input nonlinearity enhance the applicability of PFMB control scheme. Additionally, it is worth mentioning that the imperfectly matched premise membership functions are taken into account in this paper which means the number of fuzzy rules as well as membership functions for the polynomial fuzzy model and the polynomial fuzzy controller can be different. It should be noted that TSMFs are for stability analysis only and are not necessarily implemented on the polynomial fuzzy controllers. Consequently, the structure of designed polynomial fuzzy controllers will not be complicated.

II. PRELIMINARY

A. Notation

We use the following notations throughout this paper [35]. A monomial $\hat{\mathbf{x}}(t) = [\hat{x}_1(t), \hat{x}_2(t), \dots, \hat{x}_n(t)]^T$ in $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ is a function of the form $x_1^{\rho_1}(t)x_2^{\rho_2}(t)\cdots x_n^{\rho_n}(t)$, where $\rho_i \geq 0, i = 1, 2, \dots, n$, are integers. The degree of a monomial is $\rho = \sum_{i=1}^n \rho_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS if it can be written as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^m \mathbf{q}_j(\mathbf{x}(t))^2$, where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a non-zero positive integer. It can be concluded that if $\mathbf{p}(\mathbf{x}(t))$ is an SOS, $\mathbf{p}(\mathbf{x}(t)) \geq 0$. The expressions of $\mathbf{M} > 0, \mathbf{M} \geq 0, \mathbf{M} < 0$ and $\mathbf{M} \leq 0$ denote the positive, semi-positive, negative and semi-negative definite

matrices \mathbf{M} , respectively. The symbol “*” in a matrix represents the transposed element in the corresponding position and $\text{diag}\{\dots\}$ stands for a diagonal matrix.

B. Polynomial Fuzzy Model

The i^{th} rule of the polynomial fuzzy model for the nonlinear plant is given as follows [33]:

$$\begin{aligned} \text{Rule } i : & \text{ IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } \dots \text{ AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i \\ & \text{ THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t), \end{aligned} \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector of the plant; $f_\alpha(\mathbf{x}(t))$ is the premise variable corresponding to its fuzzy term M_α^i in rule i , $\alpha = 1, 2, \dots, \Psi$, and Ψ is a positive integer; $\mathbf{A}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times n}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times m}$ are the known polynomial system and input matrices, respectively; $\hat{\mathbf{x}}(t) \in \mathbb{R}^N$ is a vector of monomials in $\mathbf{x}(t)$, and it is assumed that $\hat{\mathbf{x}}(t) = \mathbf{0}$, iff $\mathbf{x}(t) = \mathbf{0}$; $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input vector. Thus, the dynamics of the system is given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^p w_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)), \quad (2)$$

where p is the number of rules in the polynomial fuzzy model; $w_i(\mathbf{x}(t))$ is the normalized grade of membership, $w_i(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(t)))}{\sum_{k=1}^p \prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{x}(t)))}$, $w_i(\mathbf{x}(t)) \geq 0, i = 1, 2, \dots, p$, and $\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1$; $\mu_{M_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\alpha = 1, 2, \dots, \Psi$, are grades of membership corresponding to fuzzy term M_α^i .

It is noted that the advantage of polynomial fuzzy models is their capability of describing nonlinear systems by fuzzy blending of polynomial subsystems. Polynomial subsystems can be handled by SOS approach and corresponding MATLAB toolbox. If the original nonlinear system is not in a polynomial form, stability conditions cannot be formulated into SOS-based conditions directly. In this case, by representing the original nonlinear system with a polynomial fuzzy model, the SOS-based stability analysis can be performed.

C. Sector Nonlinearity of control input

The input nonlinearity is given by

$$u_j(t) = \phi_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t)), \quad j = 1, 2, \dots, m, \quad (3)$$

where the control input vector is $\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_m(t)]^T \in \mathbb{R}^m$, and the control signal vector is $\tilde{\mathbf{u}}(t) = [\tilde{u}_1(t), \tilde{u}_2(t), \dots, \tilde{u}_m(t)]^T \in \mathbb{R}^m$; the control input $u_j(t)$ is assumed to have the known nonlinearity $\phi_j(\cdot)$ as shown in Fig. 1, which can be bounded by the sector $[\underline{s}_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t)), \bar{s}_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t))]$, where $\underline{s}_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t))$ and $\bar{s}_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t))$ are the lower and upper bounds, respectively. In other words, $(\phi_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t)) - \underline{s}_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t))\tilde{u}_j(t))(\phi_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t)) - \bar{s}_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t))\tilde{u}_j(t)) \leq 0$ for all j . For simplicity, we assume that the input nonlinearity (3), and the lower and upper bounds $\underline{s}_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t))$ and $\bar{s}_j(\mathbf{x}(t), \tilde{\mathbf{u}}(t))$ all pass through the origin of the plane and are all limited in the first and third quadrants. In general, however, the input nonlinearity is not necessarily bounded in the first and third quadrants.

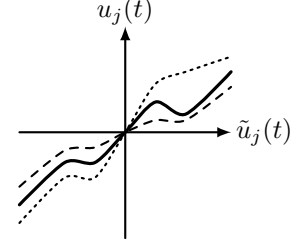


Fig. 1. Sector nonlinearity of control input. Solid line: input nonlinearity. Dashed line: lower bound of the sector. Dotted line: upper bound of the sector.

D. Polynomial Fuzzy Controller

The j^{th} rule of the polynomial fuzzy controller is given as:

$$\begin{aligned} \text{Rule } j : & \text{ IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND } \dots \text{ AND } g_\Omega(\mathbf{x}(t)) \text{ is } N_\Omega^j \\ & \text{ THEN } \tilde{\mathbf{u}}(t) = \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \end{aligned} \quad (4)$$

where $g_\beta(\mathbf{x}(t))$ is the premise variable corresponding to its fuzzy term N_β^j in rule j , $\beta = 1, 2, \dots, \Omega$, and Ω is a positive integer; $\mathbf{G}_j(\mathbf{x}(t)) \in \mathbb{R}^{m \times N}$ is the polynomial feedback gain in rule j . Thus, the following polynomial fuzzy controller is applied to the polynomial fuzzy model (2):

$$\tilde{\mathbf{u}}(t) = \sum_{j=1}^c m_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \quad (5)$$

where c is the number of rules in the polynomial fuzzy controller; $m_j(\mathbf{x}(t))$ is the normalized grade of the membership, $m_j(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(t)))}{\sum_{k=1}^c \prod_{l=1}^{\Omega} (\mu_{N_l^k}(g_l(\mathbf{x}(t))))}$, $m_j(\mathbf{x}(t)) \geq 0, j = 1, 2, \dots, c$, and $\sum_{j=1}^c m_j(\mathbf{x}(t)) = 1$; $\mu_{N_\beta^j}(g_\beta(\mathbf{x}(t)))$, $\beta = 1, 2, \dots, \Omega$, are grades of membership corresponding to the fuzzy term N_β^j .

The control objective is to stabilize the polynomial fuzzy model (2) using the polynomial fuzzy controller (5). By determining the polynomial feedback gains $\mathbf{G}_j(\mathbf{x}(t))$, we achieve $\mathbf{x}(t) \rightarrow \mathbf{0}$ as time $t \rightarrow \infty$.

III. STABILITY ANALYSIS

In this section, a general form for the approximation of membership functions is introduced first. Then more specific TSMFs are employed to implement the approximation in stability analysis. To derive stability conditions step by step, we begin by considering the PFMB control system without sector nonlinearity of control input, and then deal with the one with sector nonlinearity of control input.

A. Taylor Series Membership Function

To facilitate the stability analysis, we propose TSMFs to approximate the original membership functions. In the following analysis, for brevity, $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(\mathbf{x}(t))$ are denoted as \mathbf{x} and $\hat{\mathbf{x}}$ respectively. Let us define system states $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$, $\mathbf{x} \in \psi$, where ψ is a known bounded n dimensional state space of interest. Considering a membership function depending on $x_r, r = 1, 2, \dots, n$, we divide x_r into d_r connected substate spaces. The overall state space ψ

is then divided into η connected substate spaces which are denoted as $\psi_l, l = 1, 2, \dots, \eta$. Thus, we have the relation that $\psi = \bigcup_{l=1}^{\eta} \psi_l$ and $\eta = \prod_{r=1}^n d_r$. When x_r falls into one of the corresponding d_r divided substate spaces, we define x_{r1} and x_{r2} as the endpoints of such region. In each substate space ψ_l , we have 2^n endpoints. Totally, we have $\prod_{r=1}^n (d_r + 1)$ endpoints in overall state space ψ . In what follows, these endpoints are exploited to become operating points for approximation and even expansion points for the Taylor series. Note that the values of x_{r1} and x_{r2} are different from region to region, and are predefined. Meanwhile, membership functions are not necessarily a function of all system states.

Let us define $h_{ij}(\mathbf{x}) \equiv w_i(\mathbf{x})m_j(\mathbf{x})$, and denote the approximation of $h_{ij}(\mathbf{x})$ as $\bar{h}_{ij}(\mathbf{x})$. Therefore, the approximated membership function is defined as

$$\bar{h}_{ij}(\mathbf{x}) = \sum_{l=1}^{\eta} \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri_r l}(x_r) \delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x}) \quad \forall i, j, \quad (6)$$

where $\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x})$ is a predefined function of \mathbf{x} , demonstrating a good local approximation for the original membership function $h_{ij}(\mathbf{x})$ around an operating point $x_r = x_{ri_r}$, $r = 1, 2, \dots, n, i_r = 1, 2$, in a predefined substate space $\mathbf{x} \in \psi_l, l = 1, 2, \dots, \eta$; $v_{ri_r l}(x_r)$ exhibits the following properties: $0 \leq v_{ri_r l}(x_r) \leq 1$ and $v_{r1l}(x_r) + v_{r2l}(x_r) = 1$ for $r = 1, 2, \dots, n, i_r = 1, 2, \mathbf{x} \in \psi_l, l = 1, 2, \dots, \eta$; otherwise, $v_{ri_r l}(x_r) = 0$, leading to the property that $\sum_{l=1}^{\eta} \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri_r l}(x_r) = 1$. The function $v_{ri_r l}(x_r)$ serves as an interpolation function combining the local approximating function $\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x})$ for the approximation of the original membership function $h_{ij}(\mathbf{x})$. The approximated membership function (6) facilitates the stability analysis by bringing the local information of the original membership functions into the stability conditions.

Remark 1: $\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x})$ in (6) is a general form, which can be determined by different methods of approximation. In this paper, we employ the Taylor series expansion to approximate membership functions around the endpoints of the divided substate spaces.

The general form of the multi-variable Taylor series expansion [49] is given as follows:

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{r=1}^n (x_r - x_{r0}) \frac{\partial}{\partial x_r} \right)^k f(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_0}, \quad (7)$$

where $f(\mathbf{x})$ is an arbitrary function of \mathbf{x} ; $x_{r0}, r = 1, 2, \dots, n$, are the expansion points; $\mathbf{x}_0 = [x_{10}, x_{20}, \dots, x_{n0}]^T$; $\frac{\partial}{\partial x_r} f(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_0}$ is a constant calculated by taking the partial derivative of $f(\mathbf{x})$ and then substituting \mathbf{x} by \mathbf{x}_0 . From the Taylor series expansion (7), we substitute the expansion points by the endpoints of substate spaces to obtain $\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x})$ as:

$$\begin{aligned} \delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x}) &= \sum_{k=0}^{\lambda-1} \frac{1}{k!} \left(\sum_{r=1}^n (x_r - x_{ri_r}) \frac{\partial}{\partial x_r} \right)^k \\ &\times h_{ij}(\mathbf{x})|_{(x_r=x_{ri_r}, r=1, 2, \dots, n)} \quad \forall i, j, i_1, i_2, \dots, i_n, l, \mathbf{x} \in \psi_l, \end{aligned} \quad (8)$$

where λ is the predefined truncation order, which means the polynomial with the order of $\lambda - 1$ is applied for approximation. The TSMF is obtained by substituting (8) into (6).

B. Polynomial Fuzzy-Model-Based Control Systems

The stability of the PFMB control system without sector nonlinear of control input is investigated in this section. The input nonlinearity (3) becomes $\mathbf{u}(t) = \tilde{\mathbf{u}}(t)$. For brevity, $w_i(\mathbf{x})$ and $m_j(\mathbf{x})$ are denoted as w_i and m_j , respectively. Denoting $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N]^T$, the PFMB control system formed by the polynomial fuzzy model (2) and the polynomial fuzzy controller (5) is

$$\dot{\hat{\mathbf{x}}} = \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{T}(\mathbf{x})\dot{\mathbf{x}} = \sum_{i=1}^p w_i (\tilde{\mathbf{A}}_i(\mathbf{x})\hat{\mathbf{x}} + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{u}(t)) \quad (9)$$

$$= \sum_{i=1}^p \sum_{j=1}^c w_i m_j (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}))\hat{\mathbf{x}}, \quad (10)$$

where $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x}), \tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$, and $\mathbf{T}(\mathbf{x}) \in \mathbb{R}^{N \times n}$ is a polynomial matrix with its $(i, j)^{th}$ element defined as $T_{ij}(\mathbf{x}) = \partial \hat{x}_i(\mathbf{x}) / \partial x_j$. As $\hat{\mathbf{x}}$ is a vector of monomials in \mathbf{x} (with degree of monomial greater than 0), $\hat{\mathbf{x}} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Therefore, the system stability of (10) implies that of (2).

The following polynomial Lyapunov function candidate is employed to investigate the stability of (10):

$$V(\mathbf{x}) = \hat{\mathbf{x}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}, \quad (11)$$

where $0 < \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \mathbb{R}^{N \times N}$; $\tilde{\mathbf{x}}$ is in Remark 2.

Remark 2: To facilitate the stability analysis, define $\mathbf{K} = \{\zeta_1, \zeta_2, \dots, \zeta_s\}$ as the set of row numbers whose entries of the entire row of $\mathbf{B}_i(\mathbf{x})$ are all zeros [33], [35], and $\tilde{\mathbf{x}} = (x_{\zeta_1}, x_{\zeta_2}, \dots, x_{\zeta_s})$. Hence, we have $\frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} = \sum_{\zeta \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_{\zeta}} \sum_{i=1}^p w_i \mathbf{A}_i^{\zeta}(\mathbf{x})\hat{\mathbf{x}}$, where $\mathbf{A}_i^{\zeta}(\mathbf{x}) \in \mathbb{R}^N$ is the ζ^{th} row of $\mathbf{A}_i(\mathbf{x})$ [33].

Lemma 1: For any invertible polynomial matrix $\mathbf{X}(\mathbf{z})$ where $\mathbf{z} = [z_1, z_2, \dots, z_n]^T$, the following is true [33], [35].

$$\frac{\partial \mathbf{X}(\mathbf{z})^{-1}}{\partial z_j} = -\mathbf{X}(\mathbf{z})^{-1} \frac{\partial \mathbf{X}(\mathbf{z})}{\partial z_j} \mathbf{X}(\mathbf{z})^{-1} \quad \forall j$$

From Remark 2 and Lemma 1, we have

$$\frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} = -\mathbf{X}(\tilde{\mathbf{x}})^{-1} \left(\sum_{i=1}^p \sum_{\zeta \in \mathbf{K}} w_i \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_{\zeta}} \mathbf{A}_i^{\zeta}(\mathbf{x})\hat{\mathbf{x}} \right) \mathbf{X}(\tilde{\mathbf{x}})^{-1}. \quad (12)$$

Let us denote $h_{ij}(\mathbf{x})$ and $\bar{h}_{ij}(\mathbf{x})$ as h_{ij} and \bar{h}_{ij} , and define $\mathbf{z} = \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}$ and $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$, where $\mathbf{N}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}, j = 1, 2, \dots, c$, are arbitrary polynomial matrices. From (11), with (10) and (12), we have

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^p \sum_{j=1}^c h_{ij} \mathbf{z}^T \mathbf{Q}_{ij}(\mathbf{x}) \mathbf{z}, \quad (13)$$

where $\mathbf{Q}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T \tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{\zeta \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_{\zeta}} \mathbf{A}_i^{\zeta}(\mathbf{x})\hat{\mathbf{x}}$ for $i = 1, 2, \dots, p, j = 1, 2, \dots, c$.

Remark 3: From the Lyapunov stability theory, the asymptotic stability of (10) is guaranteed by $V(\mathbf{x}) > 0$ and $\dot{V}(\mathbf{x}) < 0$ (excluding $\mathbf{x} = \mathbf{0}$), which can be implied by $\mathbf{Q}_{ij}(\mathbf{x}) < 0$ for all i, j . However, the information of w_i and m_j is not considered, which leads to very conservative stability conditions.

In order to relax the stability conditions, in the following, some constraints of membership functions are considered such that slack polynomial matrices can be introduced in the stability analysis. Let us define the error of the approximation of membership functions as $\Delta h_{ij} = h_{ij} - \bar{h}_{ij}$. Then the lower and upper bounds of Δh_{ij} are denoted as $\underline{\gamma}_{ij}$ and $\bar{\gamma}_{ij}$, respectively. Thus, we have the property that $\underline{\gamma}_{ij} \leq \Delta h_{ij} \leq \bar{\gamma}_{ij}$, $i = 1, 2, \dots, p, j = 1, 2, \dots, c$. Based on these properties, we introduce the slack polynomial matrices $0 < \mathbf{Y}_{ij}(\mathbf{x}) = \mathbf{Y}_{ij}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$ which satisfy $\mathbf{Y}_{ij}(\mathbf{x}) \geq \mathbf{Q}_{ij}(\mathbf{x}) \forall i, j$.

Applying the above properties and slack polynomial matrices, (13) can be written as follows:

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \mathbf{z}^T \sum_{i=1}^p \sum_{j=1}^c (\bar{h}_{ij} \mathbf{Q}_{ij}(\mathbf{x}) + (\Delta h_{ij} - \underline{\gamma}_{ij} + \underline{\gamma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x})) \mathbf{z} \\ &\leq \mathbf{z}^T \sum_{i=1}^p \sum_{j=1}^c ((\bar{h}_{ij} + \underline{\gamma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}) + (\bar{\gamma}_{ij} - \underline{\gamma}_{ij}) \mathbf{Y}_{ij}(\mathbf{x})) \mathbf{z}. \end{aligned} \quad (14)$$

Moreover, we employ the properties that $\bar{h}_{ij} \geq \underline{\beta}_{ij}$ for all i, j , where $\underline{\beta}_{ij}$ is the lower bound of \bar{h}_{ij} . Then we have $(\bar{h}_{ij} - \underline{\beta}_{ij}) \mathbf{W}_{ij}(\mathbf{x}) \geq 0$, where $0 < \mathbf{W}_{ij}(\mathbf{x}) = \mathbf{W}_{ij}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$ for all i, j . Hence, (14) can be further relaxed as follows:

$$\begin{aligned} \dot{V}(\mathbf{x}) &\leq \mathbf{z}^T \sum_{i=1}^p \sum_{j=1}^c ((\bar{h}_{ij} + \underline{\gamma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}) + (\bar{\gamma}_{ij} - \underline{\gamma}_{ij}) \mathbf{Y}_{ij}(\mathbf{x}) \\ &\quad + (\bar{h}_{ij} - \underline{\beta}_{ij}) \mathbf{W}_{ij}(\mathbf{x})) \mathbf{z}. \end{aligned} \quad (15)$$

Employing TSMFs for approximating the membership functions, the approximated membership function \bar{h}_{ij} in (15) is substituted by TSMFs (6) and (8). For the fact that $\sum_{l=1}^{\eta} \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri,l}(x_r) = 1$ and $v_{ri,l}(x_r)$ is independent of rule i, j , we have

$$\begin{aligned} \dot{V}(\mathbf{x}) &\leq \mathbf{z}^T \sum_{l=1}^{\eta} \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri,l}(x_r) \sum_{i=1}^p \sum_{j=1}^c \\ &\quad ((\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x}) + \underline{\gamma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}) + (\bar{\gamma}_{ij} - \underline{\gamma}_{ij}) \mathbf{Y}_{ij}(\mathbf{x}) \\ &\quad + (\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x}) - \underline{\beta}_{ij}) \mathbf{W}_{ij}(\mathbf{x})) \mathbf{z}. \end{aligned} \quad (16)$$

The inequality $\dot{V}(\mathbf{x}) < 0$ holds if $\sum_{i=1}^p \sum_{j=1}^c ((\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x}) + \underline{\gamma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}) + (\bar{\gamma}_{ij} - \underline{\gamma}_{ij}) \mathbf{Y}_{ij}(\mathbf{x}) + (\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x}) - \underline{\beta}_{ij}) \mathbf{W}_{ij}(\mathbf{x})) < 0$ for all $i_1, i_2, \dots, i_n, l, \mathbf{x} \in \psi_l$. The above stability analysis result is summarized in the following theorem.

Theorem 1: The PFMB system (10), which is formed by the polynomial fuzzy model (2) and the polynomial fuzzy controller (5) connected in a closed loop without sector nonlinearity of control input, is guaranteed to be asymptotically stable if there exist polynomial matrices $\mathbf{Y}_{ij}(\mathbf{x}) = \mathbf{Y}_{ij}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$, $\mathbf{W}_{ij}(\mathbf{x}) = \mathbf{W}_{ij}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$, $\mathbf{N}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}$,

$i = 1, 2, \dots, p, j = 1, 2, \dots, c$, and $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \mathbb{R}^{N \times N}$ such that the following SOS-based conditions are satisfied:

$$\begin{aligned} &\nu^T (\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}}) \mathbf{I}) \nu \text{ is SOS;} \\ &\nu^T (\mathbf{Y}_{ij}(\mathbf{x}) - \varepsilon_2(\mathbf{x}) \mathbf{I}) \nu \text{ is SOS} \quad \forall i, j; \\ &\nu^T (\mathbf{Y}_{ij}(\mathbf{x}) - \mathbf{Q}_{ij}(\mathbf{x}) - \varepsilon_3(\mathbf{x}) \mathbf{I}) \nu \text{ is SOS} \quad \forall i, j; \\ &\nu^T (\mathbf{W}_{ij}(\mathbf{x}) - \varepsilon_4(\mathbf{x}) \mathbf{I}) \nu \text{ is SOS} \quad \forall i, j; \\ &-\nu^T \left(\sum_{i=1}^p \sum_{j=1}^c ((\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x}) + \underline{\gamma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}) \right. \\ &\quad \left. + (\bar{\gamma}_{ij} - \underline{\gamma}_{ij}) \mathbf{Y}_{ij}(\mathbf{x}) + (\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x}) - \underline{\beta}_{ij}) \mathbf{W}_{ij}(\mathbf{x})) \right. \\ &\quad \left. + \varepsilon_5(\mathbf{x}) \mathbf{I} \right) \nu \text{ is SOS} \quad \forall i_1, i_2, \dots, i_n, l, \mathbf{x} \in \psi_l; \end{aligned} \quad (17)$$

where $\nu \in \mathbb{R}^N$ is an arbitrary vector independent of \mathbf{x} ; $\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x})$ is a predefined function of \mathbf{x} in TSMFs (6) and (8); $\underline{\gamma}_{ij}, \bar{\gamma}_{ij}, \underline{\beta}_{ij}, i = 1, 2, \dots, p, j = 1, 2, \dots, c$, are predefined constant scalars satisfying $\Delta h_{ij} = h_{ij} - \bar{h}_{ij}, \underline{\gamma}_{ij} \leq \Delta h_{ij} \leq \bar{\gamma}_{ij}$, and $\bar{h}_{ij} \geq \underline{\beta}_{ij}$; $\varepsilon_1(\tilde{\mathbf{x}}) > 0, \varepsilon_2(\mathbf{x}) > 0, \varepsilon_3(\mathbf{x}) > 0, \varepsilon_4(\mathbf{x}) > 0$ and $\varepsilon_5(\mathbf{x}) > 0$ are predefined scalar polynomials; $\mathbf{Q}_{ij}(\mathbf{x})$ is defined in (13); all unknown polynomial matrices are calculated by SOSTOOLS; and the feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}})^{-1}, j = 1, 2, \dots, c$.

Remark 4: Referring to Theorem 1, the number of decision matrix variables is $1 + c + 2pc$, and the number of SOS conditions is $\prod_{r=1}^n (d_r + 1) + 3pc + 1$. In some cases, slack matrices $\mathbf{W}_{ij}(\mathbf{x})$ in SOS conditions (17) and (18) cannot provide less conservative results. These SOS conditions are simplified as follows: $-\nu^T \left(\sum_{i=1}^p \sum_{j=1}^c ((\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x}) + \underline{\gamma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}) + (\bar{\gamma}_{ij} - \underline{\gamma}_{ij}) \mathbf{Y}_{ij}(\mathbf{x})) + \varepsilon_5(\mathbf{x}) \mathbf{I} \right) \nu$ is SOS for all $i_1, i_2, \dots, i_n, l, \mathbf{x} \in \psi_l$. In this case, the number of variables and SOS conditions are reduced to $1 + c + pc$ and $\prod_{r=1}^n (d_r + 1) + 2pc + 1$, respectively.

C. Polynomial Fuzzy-Model-Based Control Systems with Sector Nonlinearity of Control Input

The stability of the PFMB control system with sector nonlinearity of control input is analyzed in this section. This control system is formed by the polynomial fuzzy model (2), the polynomial fuzzy controller (5), and the input nonlinearity (3). $\mathbf{u}(t)$ and $\tilde{\mathbf{u}}(t)$ are denoted as \mathbf{u} and $\tilde{\mathbf{u}}$, respectively.

In this paper, we construct a sector with nonlinear lower and upper bounds by combining local bounds using fuzzy blending to describe the input nonlinearity (3). Recalling that we aim to construct the global sector $[\underline{s}_j(\mathbf{x}, \tilde{\mathbf{u}}), \bar{s}_j(\mathbf{x}, \tilde{\mathbf{u}})]$, let us define the local sector as $[\underline{s}_{jr}(\mathbf{x}), \bar{s}_{jr}(\mathbf{x})]$ corresponding to control input u_j in the local region $\omega_r, r = 1, 2, \dots, q$, where $\omega = \bigcup_{r=1}^q \omega_r, \omega \subset \mathbb{R}^{n+m}$ are known bounded region of interest partitioned into q local regions (the partition can be achieved by the same method for ψ in Subsection III-A). $\underline{s}_{jr}(\mathbf{x})$ and $\bar{s}_{jr}(\mathbf{x})$ are the lower and upper local bounds, respectively, and they are predefined polynomials of \mathbf{x} . Note that the local bounds $\underline{s}_{jr}(\mathbf{x})$ and $\bar{s}_{jr}(\mathbf{x})$ do not depend on the control signal $\tilde{\mathbf{u}}$ due to the complexity in analysis. The global

sector $[\underline{s}_j(\mathbf{x}, \tilde{\mathbf{u}}), \bar{s}_j(\mathbf{x}, \tilde{\mathbf{u}})]$ is established by q fuzzy rules:

Rule r : IF $d_1(\mathbf{x}, \tilde{\mathbf{u}})$ is L_1^r AND \dots AND $d_\sigma(\mathbf{x}, \tilde{\mathbf{u}})$ is L_σ^r
 THEN $\underline{s}_j(\mathbf{x}, \tilde{\mathbf{u}}) = \underline{s}_{jr}(\mathbf{x}), \bar{s}_j(\mathbf{x}, \tilde{\mathbf{u}}) = \bar{s}_{jr}(\mathbf{x})$,

where r is the rule number as well as the local region number; $d_\kappa(\mathbf{x}, \tilde{\mathbf{u}})$ is the premise variable corresponding to its fuzzy term L_κ^r in rule r , $\kappa = 1, 2, \dots, \sigma$, and σ is a positive integer. Hence, the global sector $[\underline{s}_j(\mathbf{x}, \tilde{\mathbf{u}}), \bar{s}_j(\mathbf{x}, \tilde{\mathbf{u}})]$ is given by

$$\underline{s}_j(\mathbf{x}, \tilde{\mathbf{u}}) = \sum_{r=1}^q v_r(\mathbf{x}, \tilde{\mathbf{u}}) \underline{s}_{jr}(\mathbf{x}), \bar{s}_j(\mathbf{x}, \tilde{\mathbf{u}}) = \sum_{r=1}^q v_r(\mathbf{x}, \tilde{\mathbf{u}}) \bar{s}_{jr}(\mathbf{x}), \quad (19)$$

where q is predefined number of rules as well as the number of local regions; $v_r(\mathbf{x}, \tilde{\mathbf{u}})$ is a predefined membership function of rule r , which exhibits the following properties: $0 \leq v_r(\mathbf{x}, \tilde{\mathbf{u}}) \leq 1$, $r = 1, 2, \dots, q$, and $\sum_{r=1}^q v_r(\mathbf{x}, \tilde{\mathbf{u}}) = 1$. For simplicity, we use the same membership functions for all u_j , which means $v_r(\mathbf{x}, \tilde{\mathbf{u}})$ is independent of j . Consequently, the global sector for control input vector \mathbf{u} is formulated by

$$\underline{\mathbf{S}}(\mathbf{x}, \tilde{\mathbf{u}}) = \sum_{r=1}^q v_r(\mathbf{x}, \tilde{\mathbf{u}}) \underline{\mathbf{S}}_r(\mathbf{x}), \bar{\mathbf{S}}(\mathbf{x}, \tilde{\mathbf{u}}) = \sum_{r=1}^q v_r(\mathbf{x}, \tilde{\mathbf{u}}) \bar{\mathbf{S}}_r(\mathbf{x}), \quad (20)$$

where $\underline{\mathbf{S}}(\mathbf{x}, \tilde{\mathbf{u}}) = \text{diag}\{\underline{s}_1(\mathbf{x}, \tilde{\mathbf{u}}), \underline{s}_2(\mathbf{x}, \tilde{\mathbf{u}}), \dots, \underline{s}_m(\mathbf{x}, \tilde{\mathbf{u}})\}$, $\bar{\mathbf{S}}(\mathbf{x}, \tilde{\mathbf{u}}) = \text{diag}\{\bar{s}_1(\mathbf{x}, \tilde{\mathbf{u}}), \bar{s}_2(\mathbf{x}, \tilde{\mathbf{u}}), \dots, \bar{s}_m(\mathbf{x}, \tilde{\mathbf{u}})\}$, $\underline{\mathbf{S}}_r(\mathbf{x}) = \text{diag}\{\underline{s}_{1r}(\mathbf{x}), \underline{s}_{2r}(\mathbf{x}), \dots, \underline{s}_{mr}(\mathbf{x})\}$ and $\bar{\mathbf{S}}_r(\mathbf{x}) = \text{diag}\{\bar{s}_{1r}(\mathbf{x}), \bar{s}_{2r}(\mathbf{x}), \dots, \bar{s}_{mr}(\mathbf{x})\}$.

Remark 5: The local regions ω_r , the membership functions $v_r(\mathbf{x}, \tilde{\mathbf{u}})$, and the local bounds $\underline{s}_{jr}(\mathbf{x})$ and $\bar{s}_{jr}(\mathbf{x})$ are predefined. An iterative approach is provided as follows to numerically obtain the local bounds $\underline{s}_{jr}(\mathbf{x})$ and $\bar{s}_{jr}(\mathbf{x})$.

1) With predefined local regions ω_r , $r = 1, 2, \dots, q$, we compute the initial local bounds $\underline{s}_{jr}(\mathbf{x})$ and $\bar{s}_{jr}(\mathbf{x})$ in each ω_r based on the sector nonlinearity techniques [5], [34], [38].

2) With predefined membership functions such as Gaussian membership functions and triangular membership functions, we numerically check whether the global sector contains the input nonlinearity, that is $(\phi_j(\mathbf{x}, \tilde{\mathbf{u}}) - \underline{s}_j(\mathbf{x}, \tilde{\mathbf{u}})\tilde{u}_j)(\phi_j(\mathbf{x}, \tilde{\mathbf{u}}) - \bar{s}_j(\mathbf{x}, \tilde{\mathbf{u}})\tilde{u}_j) \leq 0$ for all j . The procedure terminates if these inequalities hold, otherwise go to step 3).

3) Without losing generality, it is assumed that the points along the input nonlinearity that do not satisfy the inequalities are from the transition between two adjacent local regions ω_r and ω_{r+1} . Since ω_r and ω_{r+1} are adjacent, the transition is only along one dimension z , $z \in \{x_1, \dots, x_n, u_1, \dots, u_m\}$. To ensure that the above inequalities hold, we use the safer (but more conservative) local bounds as the common bounds for both ω_r and ω_{r+1} . For instance, if $\underline{s}_{j,r}(\mathbf{x}) \leq \underline{s}_{j,r+1}(\mathbf{x})$ and $\bar{s}_{j,r}(\mathbf{x}) \leq \bar{s}_{j,r+1}(\mathbf{x})$ can be achieved, for $[\mathbf{x}^T \tilde{\mathbf{u}}^T]^T \in \omega_r \cup \omega_{r+1}$, then we use $\underline{s}_{j,r}(\mathbf{x})$ and $\bar{s}_{j,r+1}(\mathbf{x})$ as the common bounds for both ω_r and ω_{r+1} . However, for lower bounds as an example, if neither $\underline{s}_{j,r}(\mathbf{x}) \leq \underline{s}_{j,r+1}(\mathbf{x})$ nor $\underline{s}_{j,r}(\mathbf{x}) \geq \underline{s}_{j,r+1}(\mathbf{x})$ hold for $[\mathbf{x}^T \tilde{\mathbf{u}}^T]^T \in \omega_r \cup \omega_{r+1}$, go to step 4). The similar procedure is applied for upper bounds.

4) Due to the transition along dimension z , we reduce the degree of polynomials of z which exist in the local bounds

$\underline{s}_{j,r}(\mathbf{x})$, $\underline{s}_{j,r+1}(\mathbf{x})$, $\bar{s}_{j,r}(\mathbf{x})$, and $\bar{s}_{j,r+1}(\mathbf{x})$ by redesigning the local bounds using the sector nonlinearity technique [34], and then go back to step 2).

It can be found that when the degrees of polynomials of all dimension z are reduced to 0 (which is already held for dimensions $z = \{u_1, \dots, u_m\}$) by continuously executing step 4), local bounds $\underline{s}_{jr}(\mathbf{x})$ and $\bar{s}_{jr}(\mathbf{x})$ become constants, namely, \underline{s}_{jr} and \bar{s}_{jr} . After that, when all local bounds share the same bounds by continuously executing step 3), we have $\underline{s}_{j1} = \dots = \underline{s}_{jq}$ and $\bar{s}_{j1} = \dots = \bar{s}_{jq}$. As a result, the global bounds (19) become constants, for example, $\underline{s}_j(\mathbf{x}, \tilde{\mathbf{u}}) = \sum_{r=1}^q v_r(\mathbf{x}, \tilde{\mathbf{u}}) \underline{s}_{jr}(\mathbf{x}) = \sum_{r=1}^q v_r(\mathbf{x}, \tilde{\mathbf{u}}) \underline{s}_{j1} = \underline{s}_{j1}$. This is just the sector with linear bounds, which indicates that the proposed sector with nonlinear bounds varying with both system states \mathbf{x} and control signals $\tilde{\mathbf{u}}$ is more general and the above procedure can be used to obtain the generalized sector. An example of this procedure is presented in Section IV.

Since local information is brought to the construction of global sector, the proposed sector with nonlinear bounds describes the input nonlinearity more precisely. The sector with linear bounds may poorly approximate the input nonlinearity resulting in conservative stability conditions. Using the proposed nonlinear sector can achieve a better approximation to alleviate the conservativeness in the stability analysis.

For brevity, $v_r(\mathbf{x}, \tilde{\mathbf{u}})$ is denoted as v_r . The property of sector nonlinearity of control input is defined by

$$(\mathbf{u} - \sum_{r=1}^q v_r \underline{\mathbf{S}}_r(\mathbf{x}) \tilde{\mathbf{u}})^T \mathbf{\Lambda}^{-1} (\sum_{r=1}^q v_r \bar{\mathbf{S}}_r(\mathbf{x}) \tilde{\mathbf{u}} - \mathbf{u}) \geq 0, \quad (21)$$

where $\mathbf{\Lambda} = \text{diag}\{\tau_1, \dots, \tau_m\} \geq 0$ and hence $\mathbf{\Lambda}^{-1} \geq 0$.

Then some arrangement of this inequality is done before applying it to stability analysis. Expanding (21) and substituting the control signal (5), we get

$$\sum_{r_1=1}^q \sum_{r_2=1}^q \sum_{j=1}^c \sum_{k=1}^c v_{r_1} v_{r_2} m_j m_k \left[\begin{array}{c} \hat{\mathbf{x}}(\mathbf{x}(t)) \\ \mathbf{u}(t) \end{array} \right]^T \left[\begin{array}{cc} \mathbf{\Gamma}_{jkr_1r_2}^{(11)}(\mathbf{x}) & * \\ \mathbf{\Gamma}_{jkr_1r_2}^{(21)}(\mathbf{x}) & -\mathbf{\Lambda}^{-1} \end{array} \right] \left[\begin{array}{c} \hat{\mathbf{x}}(\mathbf{x}(t)) \\ \mathbf{u}(t) \end{array} \right] \geq 0, \quad (22)$$

where $\mathbf{\Gamma}_{jkr_1r_2}^{(11)}(\mathbf{x}) = -\mathbf{G}_j(\mathbf{x})^T \underline{\mathbf{S}}_{r_1}(\mathbf{x})^T \mathbf{\Lambda}^{-1} \bar{\mathbf{S}}_{r_2}(\mathbf{x}) \mathbf{G}_k(\mathbf{x})$ and $\mathbf{\Gamma}_{jkr_1r_2}^{(21)}(\mathbf{x}) = \mathbf{\Lambda}^{-1} \frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x}) + \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{2} \mathbf{G}_k(\mathbf{x})$.

The polynomial Lyapunov function candidate (11) is exploited to analyze the stability of the PFMB control system (9). Taking the derivative of (11), substituting (9) and (12) and then adding (22), we have

$$\dot{V}(\mathbf{x}) \leq \sum_{r_1=1}^q \sum_{r_2=1}^q \sum_{i=1}^p \sum_{j=1}^c \sum_{k=1}^c v_{r_1} v_{r_2} w_i m_j m_k \left[\begin{array}{c} \hat{\mathbf{x}} \\ \mathbf{u} \end{array} \right]^T \left[\begin{array}{cc} \mathbf{\Psi}_i^{(11)}(\mathbf{x}) + \mathbf{\Gamma}_{jkr_1r_2}^{(11)}(\mathbf{x}) & * \\ \mathbf{\Psi}_i^{(21)}(\mathbf{x}) + \mathbf{\Gamma}_{jkr_1r_2}^{(21)}(\mathbf{x}) & -\mathbf{\Lambda}^{-1} \end{array} \right] \left[\begin{array}{c} \hat{\mathbf{x}} \\ \mathbf{u} \end{array} \right], \quad (23)$$

where $\mathbf{\Psi}_i^{(11)}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} + \mathbf{X}(\tilde{\mathbf{x}})^{-1} \tilde{\mathbf{A}}_i(\mathbf{x}) - \mathbf{X}(\tilde{\mathbf{x}})^{-1} (\sum_{\zeta \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_\zeta} \mathbf{A}_i^\zeta(\mathbf{x}) \hat{\mathbf{x}}) \mathbf{X}(\tilde{\mathbf{x}})^{-1}$ and $\mathbf{\Psi}_i^{(21)}(\mathbf{x}) = \tilde{\mathbf{B}}_i(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1}$.

From the Lyapunov stability theory, the asymptotic stability of (9) is guaranteed by $V(\mathbf{x}) > 0$ and $\dot{V}(\mathbf{x}) < 0$ (excluding $\mathbf{x} = \mathbf{0}$), which can be implied by the following condition obtained by dropping $[\dot{\mathbf{x}}^T \ \mathbf{u}^T]^T$ and applying Schur complement to (23):

$$\begin{aligned} & \sum_{r_1=1}^q \sum_{r_2=1}^q \sum_{r_3=1}^p \sum_{r_4=1}^p \sum_{i=1}^c \sum_{l=1}^c \sum_{j=1}^c \sum_{k=1}^c v_{r_1} v_{r_2} v_{r_3} v_{r_4} w_i w_l m_j m_k \\ & (\tilde{\mathbf{A}}_i(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} + \mathbf{X}(\tilde{\mathbf{x}})^{-1} \tilde{\mathbf{A}}_i(\mathbf{x}) - \mathbf{G}_j(\mathbf{x})^T \underline{\mathbf{S}}_{r_1}(\mathbf{x})^T \mathbf{\Lambda}^{-1} \\ & \times \bar{\mathbf{S}}_{r_2}(\mathbf{x}) \mathbf{G}_k(\mathbf{x}) - \mathbf{X}(\tilde{\mathbf{x}})^{-1} \left(\sum_{\zeta \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_\zeta} \mathbf{A}_i^\zeta(\mathbf{x}) \tilde{\mathbf{x}} \right) \mathbf{X}(\tilde{\mathbf{x}})^{-1} \\ & + \mathbf{X}(\tilde{\mathbf{x}})^{-1} \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{\Lambda} \tilde{\mathbf{B}}_l(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \\ & + \mathbf{X}(\tilde{\mathbf{x}})^{-1} \tilde{\mathbf{B}}_i(\mathbf{x}) \frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x}) + \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{2} \mathbf{G}_j(\mathbf{x}) \\ & + \mathbf{G}_j(\mathbf{x})^T \frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x}) + \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{2} \tilde{\mathbf{B}}_i(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} + \mathbf{G}_j(\mathbf{x})^T \\ & \times \frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x}) + \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{2} \mathbf{\Lambda}^{-1} \frac{\underline{\mathbf{S}}_{r_3}(\mathbf{x}) + \bar{\mathbf{S}}_{r_4}(\mathbf{x})}{2} \mathbf{G}_k(\mathbf{x})) < 0. \end{aligned} \quad (24)$$

Among (24), since $\frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x})^T \mathbf{\Lambda}^{-1} \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{\underline{\mathbf{S}}_{r_1}(\mathbf{x})^T \mathbf{\Lambda}^{-1} \bar{\mathbf{S}}_{r_4}(\mathbf{x})} = \frac{\bar{\mathbf{S}}_{r_2}(\mathbf{x})^T \mathbf{\Lambda}^{-1} \underline{\mathbf{S}}_{r_3}(\mathbf{x})}{\bar{\mathbf{S}}_{r_2}(\mathbf{x})^T \mathbf{\Lambda}^{-1} \underline{\mathbf{S}}_{r_4}(\mathbf{x})}$, we combine the following two terms:

$$\begin{aligned} & -\mathbf{G}_j(\mathbf{x})^T \underline{\mathbf{S}}_{r_1}(\mathbf{x})^T \mathbf{\Lambda}^{-1} \bar{\mathbf{S}}_{r_2}(\mathbf{x}) \mathbf{G}_k(\mathbf{x}) \\ & + \mathbf{G}_j(\mathbf{x})^T \frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x}) + \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{2} \mathbf{\Lambda}^{-1} \frac{\underline{\mathbf{S}}_{r_3}(\mathbf{x}) + \bar{\mathbf{S}}_{r_4}(\mathbf{x})}{2} \mathbf{G}_k(\mathbf{x}) \\ & = \mathbf{G}_j(\mathbf{x})^T \frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x}) - \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{2} \mathbf{\Lambda}^{-1} \frac{\underline{\mathbf{S}}_{r_3}(\mathbf{x}) - \bar{\mathbf{S}}_{r_4}(\mathbf{x})}{2} \mathbf{G}_k(\mathbf{x}). \end{aligned} \quad (25)$$

Applying (25) to (24), substituting $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}})^{-1}$ and applying Schur complement, we get the following stability condition implying that (24) holds :

$$\sum_{r_1=1}^q \sum_{r_2=1}^q \sum_{i=1}^p \sum_{j=1}^c v_{r_1} v_{r_2} w_i m_j \mathbf{Q}_{ijr_1r_2}(\mathbf{x}) < 0, \quad (26)$$

where $\mathbf{Q}_{ijr_1r_2}(\mathbf{x}) = \begin{bmatrix} \Theta_{ijr_1r_2}(\mathbf{x}) & * & * \\ \mathbf{\Lambda} \tilde{\mathbf{B}}_i(\mathbf{x})^T & -\mathbf{\Lambda} & * \\ \frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x}) - \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{2} \mathbf{N}_j(\mathbf{x}) & \mathbf{0} & -\mathbf{\Lambda} \end{bmatrix}$

and $\Theta_{ijr_1r_2}(\mathbf{x}) = \mathbf{X}(\tilde{\mathbf{x}}) \tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{A}}_i(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x}) + \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{2} \mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T \frac{\underline{\mathbf{S}}_{r_1}(\mathbf{x}) + \bar{\mathbf{S}}_{r_2}(\mathbf{x})}{2} \tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{\zeta \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_\zeta} \mathbf{A}_i^\zeta(\mathbf{x}) \tilde{\mathbf{x}}$.

Remark 6: The stability condition (26) can be guaranteed by $\mathbf{Q}_{ijr_1r_2}(\mathbf{x}) < 0$ for all i, j, r_1, r_2 . However, these SOS conditions are conservative since the information of membership functions is not included. In order to relax these SOS conditions and reduce the number of conditions, TSMFs are employed and some of these conditions are grouped together.

Recalling the derivation for Theorem 1, let us define $0 < \mathbf{Y}_{ijr_1r_2}(\mathbf{x}) = \mathbf{Y}_{ijr_2r_1}(\mathbf{x})^T \in \mathfrak{R}^{(N+2m) \times (N+2m)}$, $0 < \mathbf{W}_{ijr_1r_2}(\mathbf{x}) = \mathbf{W}_{ijr_2r_1}(\mathbf{x})^T \in \mathfrak{R}^{(N+2m) \times (N+2m)}$, which satisfy $\mathbf{Y}_{ijr_1r_2}(\mathbf{x}) = \mathbf{Y}_{ijr_2r_1}(\mathbf{x}) \geq \mathbf{Q}_{ijr_1r_2}(\mathbf{x}) + \mathbf{Q}_{ijr_2r_1}(\mathbf{x})$

and $\mathbf{W}_{ijr_1r_2}(\mathbf{x}) = \mathbf{W}_{ijr_2r_1}(\mathbf{x})$ for all $i, j, r_1 \leq r_2$. The similar derivation can be obtained for (26):

$$\begin{aligned} & \sum_{r_1=1}^q \sum_{r_2=1}^q \sum_{i=1}^p \sum_{j=1}^c v_{r_1} v_{r_2} w_i m_j \mathbf{Q}_{ijr_1r_2}(\mathbf{x}) \\ & \leq \frac{1}{2} \sum_{r_1=1}^q \sum_{r_2=1}^q v_{r_1} v_{r_2} \sum_{i=1}^p \sum_{j=1}^c ((\bar{h}_{ij} + \underline{\gamma}_{ij})(\mathbf{Q}_{ijr_1r_2}(\mathbf{x}) + \mathbf{Q}_{ijr_2r_1}(\mathbf{x})) \\ & + (\bar{\gamma}_{ij} - \underline{\gamma}_{ij}) \mathbf{Y}_{ijr_1r_2}(\mathbf{x}) + (\bar{h}_{ij} - \underline{\beta}_{ij}) \mathbf{W}_{ijr_1r_2}(\mathbf{x})). \end{aligned}$$

Due to $\mathbf{Y}_{ijr_1r_2}(\mathbf{x}) = \mathbf{Y}_{ijr_2r_1}(\mathbf{x})$ and $\mathbf{W}_{ijr_1r_2}(\mathbf{x}) = \mathbf{W}_{ijr_2r_1}(\mathbf{x})$, the inequality (26) holds if $\sum_{i=1}^p \sum_{j=1}^c ((\bar{h}_{ij} + \underline{\gamma}_{ij})(\mathbf{Q}_{ijr_1r_2}(\mathbf{x}) + \mathbf{Q}_{ijr_2r_1}(\mathbf{x})) + (\bar{\gamma}_{ij} - \underline{\gamma}_{ij}) \mathbf{Y}_{ijr_1r_2}(\mathbf{x}) + (\bar{h}_{ij} - \underline{\beta}_{ij}) \mathbf{W}_{ijr_1r_2}(\mathbf{x})) < 0 \ \forall r_1 \leq r_2$. The above stability analysis result is summarized in the following theorem.

Theorem 2: The PFMB system formed by the polynomial fuzzy model (2) and the polynomial fuzzy controller (5) connected in a closed loop considering the sector nonlinearity of control input (3), is guaranteed to be asymptotically stable if there exist polynomial matrices $\mathbf{Y}_{ijr_1r_2}(\mathbf{x}) = \mathbf{Y}_{ijr_2r_1}(\mathbf{x})^T \in \mathfrak{R}^{(N+2m) \times (N+2m)}$, $\mathbf{W}_{ijr_1r_2}(\mathbf{x}) = \mathbf{W}_{ijr_2r_1}(\mathbf{x})^T \in \mathfrak{R}^{(N+2m) \times (N+2m)}$, $\mathbf{N}_j(\mathbf{x}) \in \mathfrak{R}^{m \times N}$, $i = 1, 2, \dots, p, j = 1, 2, \dots, c, r_1, r_2 = 1, 2, \dots, q, r_1 \leq r_2$, diagonal matrix $\mathbf{\Lambda} \in \mathfrak{R}^{m \times m}$, and $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \mathfrak{R}^{N \times N}$ such that the following SOS-based conditions are satisfied:

$$\begin{aligned} & \nu^T (\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}}) \mathbf{I}) \nu \text{ is SOS;} \\ & \nu^T (\mathbf{\Lambda} - \varepsilon_2(\tilde{\mathbf{x}}) \mathbf{I}) \nu \text{ is SOS;} \\ & \nu^T (\mathbf{Y}_{ijr_1r_2}(\mathbf{x}) - \varepsilon_3(\mathbf{x}) \mathbf{I}) \nu \text{ is SOS} \quad \forall i, j, r_1 \leq r_2; \\ & \nu^T (\mathbf{Y}_{ijr_1r_2}(\mathbf{x}) - (\mathbf{Q}_{ijr_1r_2}(\mathbf{x}) + \mathbf{Q}_{ijr_2r_1}(\mathbf{x})) - \varepsilon_4(\mathbf{x}) \mathbf{I}) \nu \\ & \text{ is SOS} \quad \forall i, j, r_1 \leq r_2; \\ & \nu^T (\mathbf{W}_{ijr_1r_2}(\mathbf{x}) - \varepsilon_5(\mathbf{x}) \mathbf{I}) \nu \text{ is SOS} \quad \forall i, j, r_1 \leq r_2; \\ & -\nu^T \left(\sum_{i=1}^p \sum_{j=1}^c ((\bar{h}_{ij} + \underline{\gamma}_{ij})(\mathbf{Q}_{ijr_1r_2}(\mathbf{x}) + \mathbf{Q}_{ijr_2r_1}(\mathbf{x})) \right. \\ & \left. + (\bar{\gamma}_{ij} - \underline{\gamma}_{ij}) \mathbf{Y}_{ijr_1r_2}(\mathbf{x}) + (\bar{h}_{ij} - \underline{\beta}_{ij}) \mathbf{W}_{ijr_1r_2}(\mathbf{x})) \right. \\ & \left. - \varepsilon_6(\mathbf{x}) \mathbf{I} \right) \nu \text{ is SOS} \quad \forall r_1 \leq r_2, i_1, i_2, \dots, i_n, l, \mathbf{x} \in \psi_l; \end{aligned}$$

where $\nu \in \mathfrak{R}^N$ is an arbitrary vector independent of \mathbf{x} ; $\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x})$ is a predefined function of \mathbf{x} in TSMFs (6) and (8); $\underline{\gamma}_{ij}, \bar{\gamma}_{ij}, \underline{\beta}_{ij}, i = 1, 2, \dots, p, j = 1, 2, \dots, c$, are predefined constant scalars satisfying $\Delta h_{ij} = h_{ij} - \bar{h}_{ij}, \underline{\gamma}_{ij} \leq \Delta h_{ij} \leq \bar{\gamma}_{ij}$, and $\bar{h}_{ij} \geq \underline{\beta}_{ij}$; $\varepsilon_1(\tilde{\mathbf{x}}), \varepsilon_2(\mathbf{x}), \dots, \varepsilon_6(\mathbf{x})$ are predefined positive polynomials; $\mathbf{Q}_{ijr_1r_2}(\mathbf{x})$ is defined in (26); all unknown polynomial matrices are calculated by SOSTOOLS; and the feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}})^{-1}, j = 1, 2, \dots, c$. The number of decision matrix variables is $2 + c + pc(q^2 + q)$, and the number of SOS conditions is $((q^2 + q) \prod_{r=1}^n (d_r + 1))/2 + 3pc(q^2 + q)/2 + 2$.

IV. SIMULATION EXAMPLES

In this section, two simulation examples are given to demonstrate the validity of the designed polynomial fuzzy controllers. In Example 1, we discuss the effect of TSMFs using a PFMB

control system without input nonlinearity. In Example 2, we achieve the stabilization of the inverted pendulum subject to sector nonlinearity of control input.

Example 1: (PFMB Control System) Let us consider a 3-rule polynomial fuzzy model in the form of (2) with $\hat{\mathbf{x}} = \mathbf{x} = [x_1 \ x_2]^T$. The system matrices and input matrices are:

$$\mathbf{A}_1(x_1) = \begin{bmatrix} 1.59 + 2.45x_1 & -7.29 - 0.89x_1 \\ 0.01 & -0.1 - 0.27x_1^2 \end{bmatrix},$$

$$\mathbf{A}_2(x_1) = \begin{bmatrix} 0.02 - 7.26x_1 - 0.05x_1^2 & -4.64x_1 \\ 0.35 - 0.28x_1 & -0.21 - 1.65x_1^2 \end{bmatrix},$$

$$\mathbf{A}_3(x_1) = \begin{bmatrix} -a + 0.37x_1 - 2.7x_1^2 & -4.33 - 2.73x_1^2 \\ 1.77x_1 & 0.05 - x_1^2 \end{bmatrix},$$

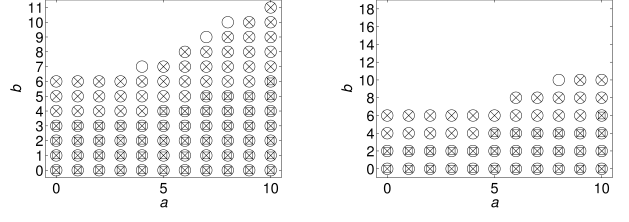
$$\mathbf{B}_1(x_1) = \begin{bmatrix} 1 + 0.37x_1 + 1.28x_1^2 \\ 0 \end{bmatrix}, \mathbf{B}_2(x_1) = \begin{bmatrix} 8 + 0.23x_1^2 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_3(x_1) = \begin{bmatrix} -b + 6 + 0.72x_1 + 1.55x_1^2 \\ -1 \end{bmatrix},$$

where a and b are constant parameters to be determined. We consider $x_1 \in [-10 \ 10]$, and the membership functions for the polynomial fuzzy model are chosen as $w_1(x_1) = 1 - 1/(1 + e^{-(x_1+4)})$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$, and $w_3(x_1) = 1/(1 + e^{-(x_1-4)})$. A polynomial fuzzy controller in the form of (5) with two rules is employed. The membership functions are chosen as $m_1(x_1) = e^{-x_1^2/12}$ and $m_2(x_1) = 1 - m_1(x_1)$.

In this example, we consider the PFMB control system without sector nonlinearity of control input. Theorem 1 is employed to design the polynomial fuzzy controller. Moreover, Theorem 1 is tested with different orders of TSMFs and densities of expansion points such that the influence to the region of stabilization can be revealed with the constant parameters a and b being chosen in the range of $0 \leq a \leq 10$ and $0 \leq b \leq 11$ at the interval of both 1.

According to the TSMFs (6) and (8), we choose the order $\lambda = 1$, and the expansion points $x_1 = \{-10, -9.5, \dots, 9.5, 10\}$. The functions $v_{11l}(x_1)$ and $v_{12l}(x_1)$ in (6) are chosen as triangular membership functions: $v_{11l}(x_1) = (x_{12} - x_1)/(x_{12} - x_{11})$, $v_{12l}(x_1) = 1 - v_{11l}(x_1)$, $\forall l$ (endpoints x_{11} and x_{12} are varied with substate space l). Based on both the original and approximated membership functions, and setting x_1 to a series of dense points, we can numerically obtain the lower and upper bounds of the error of approximation: $\underline{\gamma}_{11} = -6.9691 \times 10^{-4}$, $\underline{\gamma}_{12} = -3.9308 \times 10^{-3}$, $\underline{\gamma}_{21} = -3.8691 \times 10^{-3}$, $\underline{\gamma}_{22} = -4.8911 \times 10^{-3}$, $\underline{\gamma}_{31} = -6.9691 \times 10^{-4}$, $\underline{\gamma}_{32} = -3.9308 \times 10^{-3}$, $\bar{\gamma}_{11} = 1.7192 \times 10^{-3}$, $\bar{\gamma}_{12} = 3.3760 \times 10^{-3}$, $\bar{\gamma}_{21} = 5.9859 \times 10^{-3}$, $\bar{\gamma}_{22} = 5.5667 \times 10^{-3}$, $\bar{\gamma}_{31} = 1.7192 \times 10^{-3}$ and $\bar{\gamma}_{32} = 3.3760 \times 10^{-3}$, which satisfy $\underline{\gamma}_{ij} \leq \Delta h_{ij} \leq \bar{\gamma}_{ij}$. In this case, the simplified SOS conditions in Remark 4 are employed to replace (17) and (18) since $\delta_{ij i_1 i_2 \dots i_n l}(\mathbf{x})$ in (8) is constant and thus the slack matrices $\mathbf{W}_{ij}(\mathbf{x})$ cannot provide less conservative results. We choose $\varepsilon_1 = \dots = \varepsilon_5 = 1 \times 10^{-3}$, $\mathbf{X}(x_1)$ of degree 0, $\mathbf{Y}_{ij}(x_1)$ of degree 0, 2 and 4, and $\mathbf{N}_j(x_1)$ of degree 0 and 2. The SOS stability conditions are solved numerically by the third-party MATLAB toolbox SOSTOOLS [50]. Since higher order of Taylor series expansion may result in more terms in TSMFs, we remove the terms with the magnitude of coefficients less than 1×10^{-6} . Consequently, it could improve the efficiency of



(a) Stabilization regions.

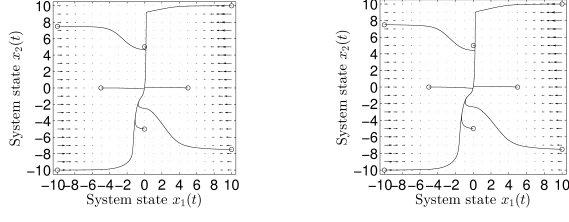
(b) Stabilization regions in new scale for comparison.

Fig. 2. Stabilization regions obtained from Theorem 1 where “□” is for $\lambda = 1$ and $x_1 = \{-10, -9.5, \dots, 9.5, 10\}$, “×” is for $\lambda = 3$ and $x_1 = \{-10, -9, \dots, 9, 10\}$, and “○” is for $\lambda = 3$ and $x_1 = \{-10, -9.5, \dots, 9.5, 10\}$.

the SOSTOOLS searching for a feasible solution. It should be noted that the lower and upper bounds of the approximation errors have taken into account the polynomial terms with magnitude of coefficients less than 1×10^{-6} removed from the TSMFs. As shown in Fig. 2(a), the stabilization region is indicated by “□” (Case 1).

To investigate the influence of the order of TSMFs, we increase the order to $\lambda = 3$ and choose the expansion points $x_1 = \{-10, -9, \dots, 9, 10\}$. With functions $v_{11l}(x_1)$ and $v_{12l}(x_1)$ in the same form as Case 1, the lower and upper bounds of approximation error are found numerically that $\underline{\gamma}_{11} = -8.5507 \times 10^{-4}$, $\underline{\gamma}_{12} = -2.5989 \times 10^{-3}$, $\underline{\gamma}_{21} = -1.9905 \times 10^{-3}$, $\underline{\gamma}_{22} = -2.6499 \times 10^{-3}$, $\underline{\gamma}_{31} = -8.5507 \times 10^{-4}$, $\underline{\gamma}_{32} = -2.5989 \times 10^{-3}$, $\bar{\gamma}_{11} = 7.3180 \times 10^{-4}$, $\bar{\gamma}_{12} = 2.2798 \times 10^{-3}$, $\bar{\gamma}_{21} = 2.0871 \times 10^{-3}$, $\bar{\gamma}_{22} = 2.6081 \times 10^{-3}$, $\bar{\gamma}_{31} = 7.3180 \times 10^{-4}$ and $\bar{\gamma}_{32} = 2.2798 \times 10^{-3}$. In this case, the lower bound of \bar{h}_{ij} is obtained that $\underline{\beta}_{11} = -3.0938 \times 10^{-5}$, $\underline{\beta}_{12} = -4.0676 \times 10^{-5}$, $\underline{\beta}_{21} = 5.9414 \times 10^{-7}$, $\underline{\beta}_{22} = -4.0687 \times 10^{-4}$, $\underline{\beta}_{31} = -3.0938 \times 10^{-5}$ and $\underline{\beta}_{32} = -4.0676 \times 10^{-5}$. In this case, SOS conditions in Theorem 1 are employed. We choose $\mathbf{W}_{ij}(x_1)$ of degree 4 and keep other parameters and settings the same as in Case 1. The stabilization region is indicated by “×” (Case 2) in Fig. 2(a), which shows that the higher the order is, the larger the stabilization region can be achieved. It should be noted that although lower density of expansion points is employed compared with Case 1, the higher order of polynomials for the approximation of membership functions generally achieves smaller approximation error, playing a role for achieving a larger size of stabilization region.

To show the effect of denser expansion points to the stabilization region, we choose a set of more intensive points $x_1 = \{-10, -9.5, \dots, 9.5, 10\}$. With the same order $\lambda = 3$ and functions $v_{11l}(x_1)$ and $v_{12l}(x_1)$, the lower and upper bounds of approximation error can be obtained numerically that $\underline{\gamma}_{11} = -1.0235 \times 10^{-4}$, $\underline{\gamma}_{12} = -3.1486 \times 10^{-4}$, $\underline{\gamma}_{21} = -2.5744 \times 10^{-4}$, $\underline{\gamma}_{22} = -3.2767 \times 10^{-4}$, $\underline{\gamma}_{31} = -1.0235 \times 10^{-4}$, $\underline{\gamma}_{32} = -3.1486 \times 10^{-4}$, $\bar{\gamma}_{11} = 1.0770 \times 10^{-4}$, $\bar{\gamma}_{12} = 3.0990 \times 10^{-4}$, $\bar{\gamma}_{21} = 2.5673 \times 10^{-4}$, $\bar{\gamma}_{22} = 3.2669 \times 10^{-4}$, $\bar{\gamma}_{31} = 1.0770 \times 10^{-4}$ and $\bar{\gamma}_{32} = 3.0990 \times 10^{-4}$. The lower bound of \bar{h}_{ij} is acquired that $\underline{\beta}_{11} = -3.0938 \times 10^{-5}$, $\underline{\beta}_{12} = -6.0276 \times 10^{-5}$, $\underline{\beta}_{21} = 5.9414 \times 10^{-7}$, $\underline{\beta}_{22} = -8.7253 \times 10^{-6}$, $\underline{\beta}_{31} = -3.0938 \times 10^{-5}$ and $\underline{\beta}_{32} = -6.0276 \times$



(a) Case 2 in Example 1 with $a = 8$ and $b = 9$. (b) Case 3 in Example 1 with $a = 8$ and $b = 10$.

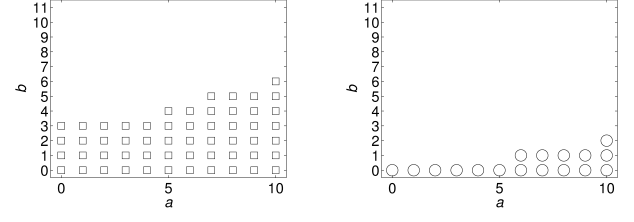
Fig. 3. Behavior in $x_1 - x_2$ plane.

10^{-5} . Keeping other settings the same as in Case 2, we obtain the stabilization region indicated by “○” (Case 3) in Fig. 2(a). It demonstrates that denser expansion points lead to a larger stabilization region since the approximation error is reduced.

In this example, it can be concluded that more intensive expansion points and higher order of TSMFs result in a larger stabilization region. The reason is that more information of membership functions is included in stability conditions. However, the computational demand increases meanwhile.

For verification, we show the phase plots of system states corresponding to Cases 2 and 3 with initial conditions indicated by “○” as shown in Fig. 3. With $a = 8$ and $b = 9$ in Case 2, the polynomial feedback gains are obtained that $\mathbf{G}_1(x_1) = [-7.8891 \times 10^{-1}x_1^2 - 1.2665 - 3.8923 \times 10^{-1}x_1^2 + 1.8943 \times 10^{-1}]$ and $\mathbf{G}_2(x_1) = [-4.9509 \times 10^{-1}x_1^2 - 2.0497 - 2.3899 \times 10^{-1}x_1^2 + 6.6710 \times 10^{-1}]$. With $a = 8$ and $b = 10$ in Case 3, the polynomial feedback gains are obtained that $\mathbf{G}_1(x_1) = [-6.3270 \times 10^{-1}x_1^2 - 1.1512 - 3.4070 \times 10^{-1}x_1^2 + 2.1688 \times 10^{-1}]$ and $\mathbf{G}_2(x_1) = [-3.1869 \times 10^{-1}x_1^2 - 1.7261 - 1.6124 \times 10^{-1}x_1^2 + 7.1676 \times 10^{-1}]$.

To show the merits of the proposed imperfectly matched premises approach in this paper, we compare the stabilization regions in Fig. 2(a) with the three results obtained by the stability conditions in Remark 3, Remark 4 and [24]. For making a fair comparison, we set the same parameters as those in Example 1 when possible. For the SOS-based stability conditions in Remark 3 which do not consider any information of membership functions, there is no stabilization region found within the range of $0 \leq a \leq 10$ and $0 \leq b \leq 11$. With respect to Remark 4, the slack matrices $\mathbf{W}_{ij}(\mathbf{x})$ are not employed in the stability conditions. Comparing Fig. 4(a) with Fig. 2(a), although the introduction of $\mathbf{W}_{ij}(\mathbf{x})$ does not improve the results for Case 1, it has great improvement on Cases 2 and 3. The reason is that Case 1 only involves constant $\delta_{ij i_1 i_2 \dots i_{n_l}}(\mathbf{x})$ in (18) which is invariant with system states. When the order of TSMF becomes larger, $\delta_{ij i_1 i_2 \dots i_{n_l}}(\mathbf{x})$ becomes variant with system states, and *Positivstellensatz* multipliers are needed to bring local information to reduce the conservativeness [34]. With *Positivstellensatz* multipliers $\mathbf{W}_{ij}(\mathbf{x})$, the advantages of higher-order TSMFs resulting in smaller approximation error (Cases 2 and 3 in Fig. 2(a)) can be demonstrated. It is worthy to mention that even though slack matrices $\mathbf{W}_{ij}(\mathbf{x})$ are not applied, results of Case 1 where TSMFs are employed still outperforms the results from Remark 3 without TSMFs, which proves that TSMFs bringing information of membership functions into stability conditions lead to relaxed stability



(a) Stabilization regions obtained from Remark 4 from [24] where “○” is for $x_1 = -10, -9.5, \dots, 9.5, 10$. (b) Stabilization regions obtained from Remark 4 from [24] where “□” is for $\lambda = 1$ and $\{-10, -9.5, \dots, 9.5, 10\}$.

Fig. 4. Stabilization regions for comparison.

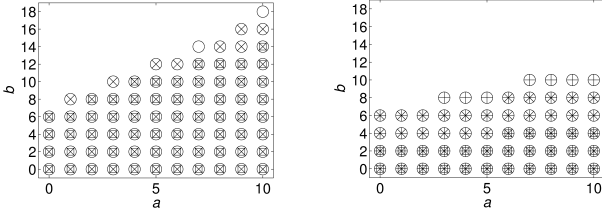
conditions. Note that slack matrices $\mathbf{Y}_{ij}(\mathbf{x})$ are used to cast off the error term Δh_{ij} and thus cannot be removed like $\mathbf{W}_{ij}(\mathbf{x})$. Results given by the SOS-based stability conditions in [24] are shown in Fig. 4(b). It can be concluded that the stabilization region obtained by the method in this paper is larger suggesting that the proposed SOS-based stability conditions are more relaxed. Hence, the conservativeness of proposed stability conditions is less than previous ones.

In the following, we compare the existing PDC techniques with proposed imperfectly matched premises approach. It should be noted that the PFMB control system considered in this example are with different sets of membership functions for polynomial fuzzy model and polynomial fuzzy controller. The existing PDC SOS-based stability conditions [6]–[8], [33] in general cannot be applied. To achieve the comparison, we have to consider a special case where the membership functions and the number of the fuzzy controller are the same as those of the fuzzy model such that the existing PDC results can be applied.

We consider the nonlinear model in Example 1 for comparisons. The only difference is that the membership functions of the fuzzy controller become $m_1(x_1) = 1 - 1/(1 + e^{-(x_1+4)})$, $m_2(x_1) = 1 - m_1(x_1) - m_3(x_1)$, and $m_3(x_1) = 1/(1 + e^{-(x_1-4)})$ (these are the membership functions of the fuzzy model as well) such that $m_i(x_1) = w_i(x_1)$ and $c = p$.

We then rearrange Fig. 2(a) into a new scale (for x and y axes) for easy comparison as shown in Fig. 2(b) such that all figures shown below will have the same scales. Fig. 2(b) shows the stabilization region with the membership functions of the fuzzy controller as $m_1(x_1) = e^{-x_1^2/12}$ and $m_2(x_1) = 1 - m_1(x_1)$. Then under the PDC design, the stabilization regions obtained from Theorem 1 are shown in Fig. 5(a). Results from existing papers for PDC design are shown in Fig. 5(b). To achieve a fair comparison, when generating Fig. 5(b), we consider the same settings as in Example 1 by choosing polynomial feedback gains of degree 0 and 2 in x_1 , and slack matrices of degree 0, 2 and 4 in x_1 . It should be noted that the analysis results from the existing papers are LMI-based. We have extended their analysis results to SOS-based.

Comparing with Fig. 5(a) and Fig. 5(b), the proposed imperfectly matched premises approach provides less conservative results than existing PDC design approaches in terms of the size of stabilization regions. This is due to the contribution of the approximated membership functions which make the



(a) Stabilization regions under PDC design ($m_i(\mathbf{x}) = w_i(\mathbf{x})$ and design ($m_i(\mathbf{x}) = w_i(\mathbf{x})$ and $c = p$) obtained from Theorem obtained from existing papers where 1 where “□” is for $\lambda = 1$ and “□” is for Theorem 2 in [33]; “x” is for $\lambda = 3$ and Theorem 5 in [7]; and “+” is for $\lambda = 3$ and fuzzy summations in [8]. $x_1 = \{-10, -9.5, \dots, 9.5, 10\}$, is for Theorem 1 in [6]; “○” is for $x_1 = \{-10, -9, \dots, 9, 10\}$, Theorem 5 ($n = 4$ is the order of and “○” is for $\lambda = 3$ and fuzzy summations) in [8]. $x_1 = \{-10, -9.5, \dots, 9.5, 10\}$.

Fig. 5. Stabilization regions for comparison.

stability conditions membership-function-dependent.

It should be noted that the existing the PDC design approaches cannot be applied to the imperfectly matched premises case. However, our proposed approach provides a more effective treatment to imperfectly matched premises case where the stabilization regions are shown in Fig. 2(b). Comparing with Fig. 2(b) and Fig. 5(b), it can be seen that our stabilization regions obtained from imperfectly matched premises case are comparable to PDC case.

Example 2: (PFMB Control System with Sector Nonlinearity of Control Input) In this example, an inverted pendulum on a cart [3] is modeled by the PFMB control system, and the sector nonlinearity of control input is imposed on the system. The state space equation of the inverted pendulum is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g \sin(x_1) - am_p L x_2^2 \sin(x_1) \cos(x_1) - a \cos(x_1) u}{4L/3 - am_p L \cos^2(x_1)} \end{aligned} \quad (27)$$

where $\mathbf{x} = [x_1 \ x_2]^T$ is the state vector; $g = 9.8m/s^2$ is the acceleration of gravity; $m_p = 2kg$ and $M_c = 8kg$ are the mass of the pendulum and the cart, respectively; $a = 1/(m_p + M_c)$; $2L = 1m$ is the length of the pendulum; and u is the control input force imposed on the cart.

According to the state space form, we construct the polynomial fuzzy model for the inverted pendulum. Let us define the region of interest as $x_1 \in [-\frac{75\pi}{180}, \frac{75\pi}{180}]$. Within the bounded region of interest, we represent the nonlinear term $f_1(x_1) = \frac{\cos(x_1)}{4L/3 - am_p L \cos^2(x_1)}$ by sector nonlinearity technique [34] as follows: $f_1(x_1) = \mu_{M_1^1}(x_1)f_{1_{min}} + \mu_{M_1^2}(x_1)f_{1_{max}}$, where $\mu_{M_1^1}(x_1) = \frac{f_1(x_1) - f_{1_{max}}}{f_{1_{min}} - f_{1_{max}}}$, $\mu_{M_1^2}(x_1) = 1 - \mu_{M_1^1}(x_1)$, $f_{1_{min}} = 0.3922$, $f_{1_{max}} = 1.7647$. In order to reduce the number of rules and computational burden, other nonlinear terms $\sin(x_1)$ and $\tan(x_1)$ are approximated by polynomials: $\sin(x_1) \approx s_1 x_1$ and $\tan(x_1) \approx t_1 x_1$, where $s_1 = 0.8386$ and $t_1 = 1.7336$. Consequently, the inverted pendulum is formulated by a 2-rule polynomial fuzzy model with the following system and input matrices, $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \ x_2]^T$, $\mathbf{A}_1(x_2) = \begin{bmatrix} 0 & 1 \\ a_1(x_2) & 0 \end{bmatrix}$,

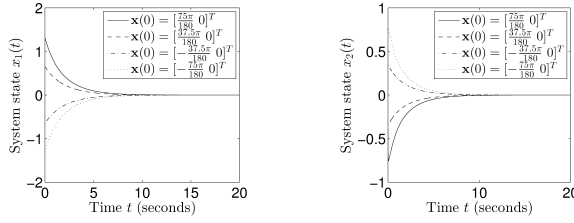
$$\mathbf{A}_2(x_2) = \begin{bmatrix} 0 & 1 \\ a_2(x_2) & 0 \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 0 \\ -f_{1_{min}} a \end{bmatrix}, \text{ and } \mathbf{B}_2 = \begin{bmatrix} 0 \\ -f_{1_{max}} a \end{bmatrix}, \text{ where } a_1(x_2) = f_{1_{min}}(gt_1 - am_p L x_2^2 s_1), a_2(x_2) = f_{1_{max}}(gt_1 - am_p L x_2^2 s_1). \text{ The membership functions are } w_1(x_1) = \mu_{M_1^1}(x_1) \text{ and } w_2(x_1) = \mu_{M_1^2}(x_1). \text{ Without losing generality, we use the same membership functions for fuzzy controllers } (m_1(x_1) = w_1(x_1) \text{ and } m_2(x_1) = w_2(x_1)).$$

Based on the polynomial fuzzy model of inverted pendulum, we consider the sector nonlinearity of control input (3). Let us define the input nonlinearity as $u = \tilde{u}(1 - 0.15 \sin(0.001\tilde{u}))$ [40], where \tilde{u} is the control signal. With bounded region of interest $\tilde{u} \in [-4000, 4000]$, we construct the nonlinear lower and upper bounds of the sector by combining bounds of three local regions $[-4000, -1333]$, $[-1333, 1333]$, and $[1333, 4000]$. The membership functions for the above regions are $v_1(\tilde{u}) = 1 - 1/(1 + e^{-(\tilde{u}+2000)/200})$, $v_2(\tilde{u}) = 1 - v_1(\tilde{u}) - v_3(\tilde{u})$, and $v_3(\tilde{u}) = 1/(1 + e^{-(\tilde{u}-2000)/200})$, respectively.

Remark 5 is employed to find lower and upper bounds for these three local regions. Step 1), based on sector nonlinearity technique [34], initial local bounds are obtained as $\underline{S}_1 = 8.8648 \times 10^{-1}$, $\underline{S}_2 = 8.5423 \times 10^{-1}$, $\underline{S}_3 = 8.5000 \times 10^{-1}$, $\bar{S}_1 = 1.1500$, $\bar{S}_2 = 1.1458$, and $\bar{S}_3 = 1.1135$. Step 2), with predefined membership functions, the established sector does not contain the input nonlinearity, and thus go to Step 3). Step 3), for lower bounds, those points of input nonlinearity outside the established sector are from the transition between local regions $[-1333, 1333]$ and $[1333, 4000]$. Due to $\underline{S}_3 < \underline{S}_2$, we take \underline{S}_3 as the common lower bound for both regions $[-1333, 1333]$ and $[1333, 4000]$ which means $\underline{S}_2 = \underline{S}_3 = 8.5000 \times 10^{-1}$. For upper bounds, those points along input nonlinearity which are not within the sector come from the transition between regions $[-4000, -1333]$ and $[-1333, 1333]$. Due to $\bar{S}_1 > \bar{S}_2$, we take \bar{S}_1 as the common upper bound for both regions $[-4000, -1333]$ and $[-1333, 1333]$ meaning $\bar{S}_1 = \bar{S}_2 = 1.1500$. Back to Step 2), the adjusted sector contains the input nonlinearity, and the procedure terminates with $\underline{S}_1 = 8.8648 \times 10^{-1}$, $\underline{S}_2 = \underline{S}_3 = 8.5000 \times 10^{-1}$, $\bar{S}_1 = \bar{S}_2 = 1.1500$, and $\bar{S}_3 = 1.1135$.

Therefore, the nonlinear lower and upper bounds of the sector are defined as in (20). For comparison purposes, conventional linear bounds of the sector are obtained by the sector nonlinearity technique to describe the input nonlinearity and facilitate the stability analysis. Compared with the sector with linear bounds $[0.85, 1.15]$, the proposed sector with nonlinear bounds has smaller area which is included by the sector with linear bounds. As a result, the proposed sector describes the input nonlinearity more accurately and less conservatively.

Theorem 2 is employed to achieve the stabilization of the inverted pendulum. We choose the order $\lambda = 1$, the expansion points $x_1 = \{-\frac{75\pi}{180}, -\frac{60\pi}{180}, \dots, \frac{60\pi}{180}, \frac{75\pi}{180}\}$, and $v_{11l}(x_1) = (x_{12} - x_1)/(x_{12} - x_{11})$, $v_{12l}(x_1) = 1 - v_{11l}(x_1)$, $\forall l$. Thus, the lower and upper bounds of the approximation error and the lower bound of \bar{h}_{ij} are obtained as follows: $\gamma_{11} = -2.1410 \times 10^{-2}$, $\gamma_{12} = -1.2875 \times 10^{-2}$, $\gamma_{21} = -1.2875 \times 10^{-2}$, $\gamma_{22} = -1.9630 \times 10^{-2}$, $\bar{\gamma}_{11} = 7.6578 \times 10^{-13}$, $\bar{\gamma}_{12} = 2.0519 \times 10^{-2}$, $\bar{\gamma}_{21} = 2.0519 \times 10^{-2}$, $\bar{\gamma}_{22} = 2.7206 \times 10^{-2}$, $\beta_{11} = \dots = \beta_{22} = 0$.



(a) Time response of system state x_1 . (b) Time response of system state x_2 .

Fig. 6. Time response of system state x_1 and x_2 of the inverted pendulum with 4 different initial conditions.

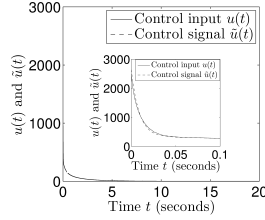


Fig. 7. Time response of control input u and control signal \tilde{u} under the initiation condition $\mathbf{x}(0) = [\frac{75\pi}{180} \ 0]^T$.

In order to reduce the computational demand, slack matrices $\mathbf{Y}_{ijr_1r_2}(\mathbf{x})$ and $\mathbf{W}_{ijr_1r_2}(\mathbf{x})$ are assumed to be identical for all r_1, r_2 , that is, $\mathbf{Y}_{ijr_1r_2}(\mathbf{x}) = \mathbf{Y}_{ij}(\mathbf{x})$, $\mathbf{W}_{ijr_1r_2}(\mathbf{x}) = \mathbf{W}_{ij}(\mathbf{x}) \ \forall r_1, r_2$. We choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 1 \times 10^{-7}$, $\varepsilon_4 = \varepsilon_6 = 1 \times 10^{-8}$, \mathbf{X} of degree 0, $\mathbf{Y}_{ij}(x_2)$, $\mathbf{W}_{ij}(\mathbf{x})$, and $\mathbf{\Lambda}(x_2)$ of degree 2, and $\mathbf{N}_j(x_2)$ of degree 1 ($\mathbf{W}_{ij}(\mathbf{x})$ has monomials concerning both x_1 and x_2). Then we can obtain the feedback gains $\mathbf{G}_1 = [2.0234 \times 10^3 \ 2.9367 \times 10^3]$ and $\mathbf{G}_2 = [1.5256 \times 10^3 \ 2.9148 \times 10^3]$. Note that since we remove the terms with the magnitude of coefficients less than 1×10^{-6} , the polynomial feedback gains $\mathbf{G}_1(x_2)$ and $\mathbf{G}_2(x_2)$ are reduced to constant feedback gains \mathbf{G}_1 and \mathbf{G}_2 .

As can be seen from Fig. 6 and 7, the inverted pendulum is asymptotically stable, and the control input varies with the original control signal.

V. CONCLUSION

The stability of PFMB control system has been investigated using SOS approach. To relax the stability conditions, TSMFs have been proposed to approximate the original membership functions. The SOS stability conditions are progressively less conservative by increasing the order of TSMFs and density of expansion points. Additionally, the sector nonlinearity of control input has been considered in the stability analysis. The bounds of the sector depending on both system states and control signal are allowed to be nonlinear. Simulation examples have been given to prove the effectiveness of the proposed method.

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H.K. Lam (M’98-SM’10) received the B.Eng. (Hons.) and Ph.D. degrees from the Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hong Kong, in 1995 and 2000, respectively.

From 2000 to 2005, he was a Postdoctoral Fellow and a Research Fellow with the Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, respectively. In 2005, he joined Kings College London, London, U.K., as a Lecturer and currently is Reader.



Chuang Liu received the BEng degree in Mechanical Engineering from Tsinghua University, Beijing, China, in 2011, and the MSc degree in Robotics from King’s College London, London, U.K., in 2013.

He is currently a PhD student at King’s College London. His research interests include fuzzy-model-based control and its applications.



Ligang Wu (M’10SM’12) received the B.S. degree in Automation from Harbin University of Science and Technology, China in 2001; the M.E. degree in Navigation Guidance and Control from Harbin Institute of Technology, China in 2003; the PhD degree in Control Theory and Control Engineering from Harbin Institute of Technology, China in 2006. From January 2006 to April 2007, he was a Research Associate in the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong. From September 2007 to June 2008, he was a

Senior Research Associate in the Department of Mathematics, City University of Hong Kong, Hong Kong. From December 2012 to December 2013, he was a Research Associate in the Department of Electrical and Electronic Engineering, Imperial College London, London, UK. In 2008, he joined the Harbin Institute of Technology, China, as an Associate Professor, and was then promoted to a Professor in 2012.



Xudong Zhao was born in Harbin, China, on July, 7, 1982. He received the B.S. degree in Automation from Harbin Institute of Technology in 2005 and the Ph.D. degree from Control Science and Engineering from Space Control and Inertial Technology Center, Harbin Institute of Technology in 2010.

Dr. Zhao was a lecturer and an associate professor at the China University of Petroleum, China. Since 2013, he joined Bohai University, China, where he is currently a professor.